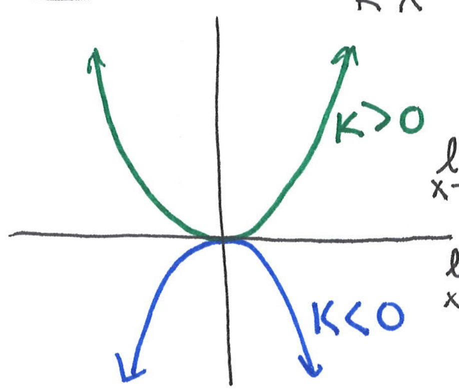
Lecture 11/28/23 Short Run behavior

Review:

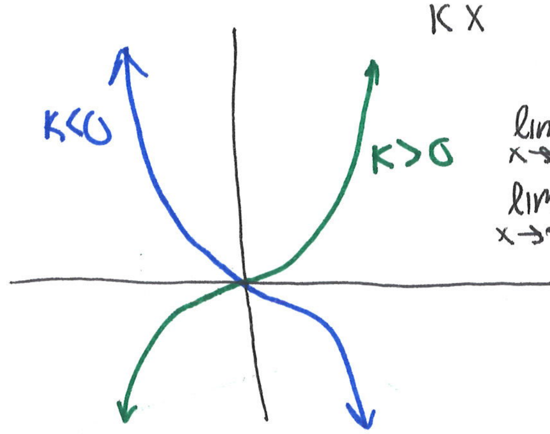
$Kx^{\text{pos. even}}$



$\lim_{x\to\infty} f(x) = \infty$, $-\infty$

$\lim_{x\to-\infty} f(x) = \infty$, $-\infty$

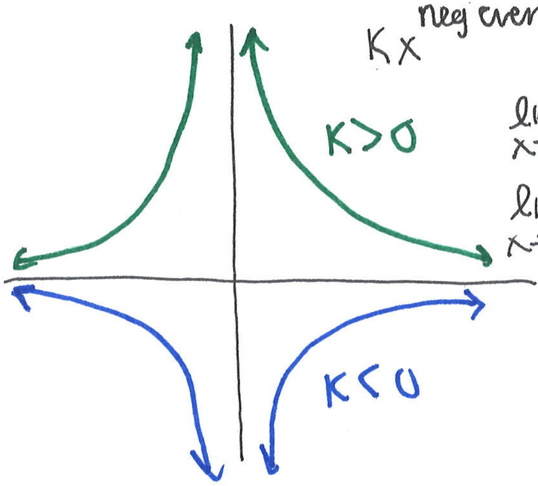
$Kx^{\text{pos odd}}$



$\lim_{x\to\infty} f(x) = \infty$, $-\infty$

$\lim_{x\to-\infty} f(x) = -\infty$, ∞

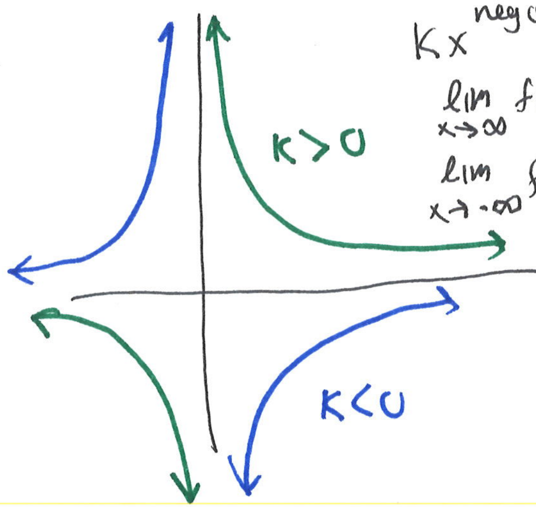
$Kx^{\text{neg even}}$



$\lim_{x\to\infty} f(x) = 0$, 0

$\lim_{x\to-\infty} f(x) = 0$, 0

$Kx^{\text{neg odd}}$



$\lim_{x\to\infty} f(x) = 0$, 0

$\lim_{x\to-\infty} f(x) = 0$, 0

Recall: the long run behavior of a polynomial $a_nx^n + \cdots + a_1x + a_0$ is the same as the long run behavior for its leading term a_nx^n.

#1: Find the roots of $2x^2 - 14x$ and state its long run behavior.

Roots: $0 = 2x^2 - 14x = (2x)(x-7)$ roots are $x = 0, 7$

L.R.B: Need to find long run behavior of $2x^2$ since that's
$2x^2 - 14x$ leading term.

$$\boxed{\lim_{x\to\infty} 2x^2 = \infty \qquad \lim_{x\to-\infty} 2x^2 = \infty}$$

Long run behavior tells us what happens far away; we will now study what happens "closer" & i.e short run behavior. This involves looking at roots!

Short run behavior:

Fundamental Thm of Algebra: Every polynomial can be factored into linear terms with complex numbers

$$p(x) = K(x-r_1)\cdots(x-r_n)$$

r_i may be complex (we will not worry about that)

~~Some roots may occur more than one.~~

Ex:

$$4x^2-49 = (2x-7)(2x+7)$$

$$x^3-7x^2 = x^2(x-7)$$

Caution some roots come up more than once. We will give this a name!!

Defn: let $p(x) = k(x-r_1)^{\alpha_1}\cdots(x-r_n)^{\alpha_n}$ where $r_i \neq r_j$ if $i \neq j$. I.e no r_i is the same. We say $p(x)$ has a root at r_i of multiplicity α_i.

Ex: $F(x) = (x+3)^2(x-2)(x-4)^2$

root of -3 mult. 2

root of 2 mult. 1

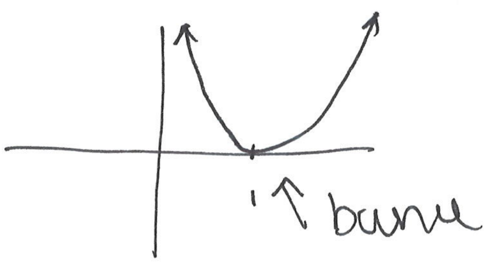
root of 4 mult. 2.

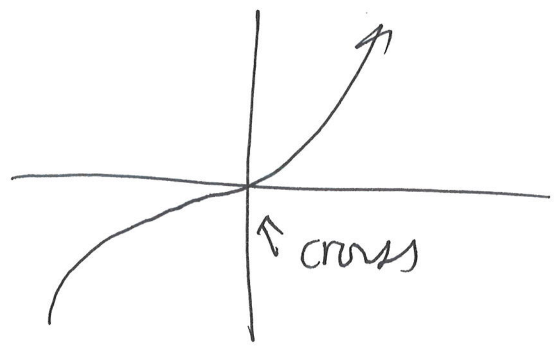
we can tell what a polynomial looks like at roots if we knew its multiplicity; (this is what we call short run behavior).

let $f(x)$ be a polynomial function

1) A a root of even mult. (2, 4, 6, 8, ...) bounces off x-axis
2) A root of odd mult. (1, 3, 5, 7, ...) crosses x-axis

Ex $(x-1)^2$ $\qquad\qquad$ x^3





Ex: lets analyse l.r.b and s.r.b of $f(x) = (x+3)^2(x-2)(x-4)^2$

lead term	degree	zeros w/ multiplicities	"bounces"	"crosses"	l.r.b $x \to \infty$	l.r.b $x \to -\infty$
x^5	5	$-3\ (2),\ 2\ (1),\ 4\ (2)$	$x=-3$ $x=4$	$x=2$	∞	$-\infty$

multiply $x^2 \cdot x \cdot x^2$

$(x+3)^2\ (x-2)\ (x-4)^2$

Ex: $\quad k(x) = 2(x+1)(x^2-9)$ Graph!

① S.R.B

Factor $f(x) = 2(x+1)(x^2-9) = 2(x+1)(x-3)(x+3)$

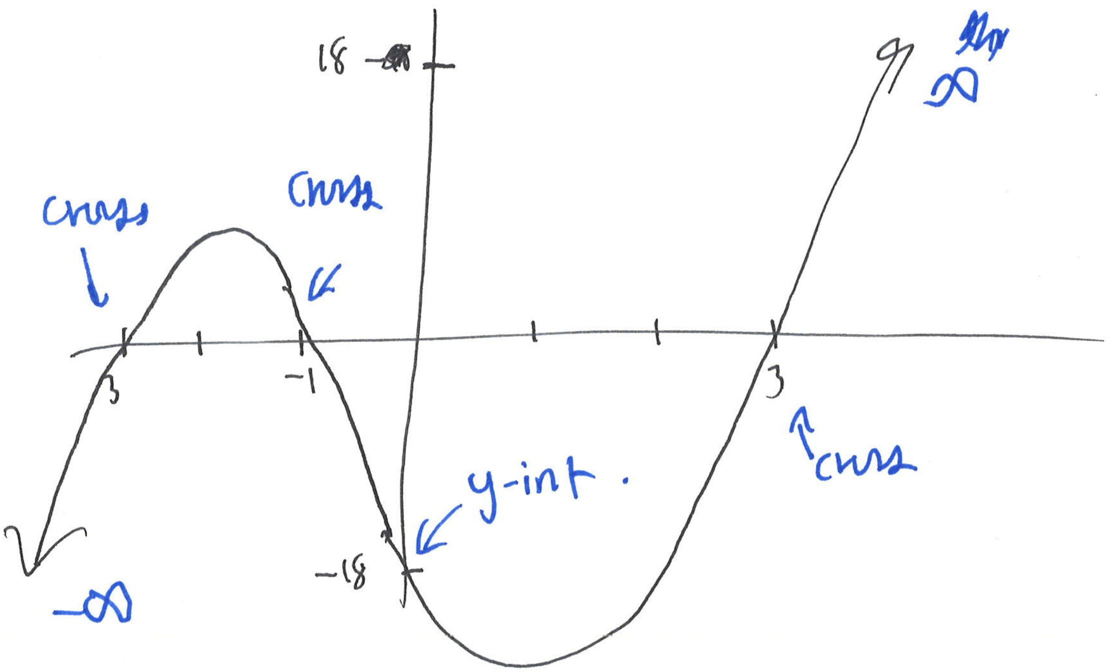
Cross at $x = -1, 3, -3$

2 LRB:

- Find leading term: $2x^3$ (check!)

$\lim_{x \to \infty} K(x) = \infty$ $\lim_{x \to -\infty} K(x) = -\infty$

3 y-int: Plug in $x=0$ $(0, -18)$



#6 (1) Find roots; $x=-1$ $x=1$ cross ($x=3$)
determine possible mult. bounce bounce

smallest possible choice

(2) $f(x) = K(x+1)^{2}(x-1)^2(x-3)$

(3) Find K using ~~...~~ any point on graph:
(2,1) is on graph so

$1 = K(2+1)^2(2-1)^2(2-3)$
$1 = K\, 9 \cdot 1 \cdot (-1)$
$K = \frac{-1}{9}$

So $f(x) = -\frac{1}{9}(x+1)^2(x-1)^2(x-3)$

Lecture: 11/20/23: Algebraic Functions

Defn: An algebraic function is

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. Some (most) call these rational functions.

Ex:

$$\frac{x^2-1}{x^4+1+x^2}$$

Reducing Rational Functions:

Ex:

$$\frac{(x-4)(2x+1)}{(x-3)(x+3)(x-4)} = \boxed{\frac{2x+1}{(x-3)(x+3)}}$$

Ex:

$$\frac{9x^2}{15x^8} = \boxed{\frac{3}{5x^6}}$$

Add/Sub/Mult/Divide Rational Functions:

This works the same way as fractions of numbers.

Ex:

$$\frac{1}{x^2-1} - \frac{1}{x^2+3x+2} = \frac{x^2+3x+2}{(x^2-1)(x^2+3x+2)} - \frac{x^2-1}{(x^2+3x+2)(x^2-1)}$$

$$= \boxed{\frac{3x+1}{(x^2-1)(x^2+3x+2)}}$$

Ex:

$$\frac{2}{(x+5)(x+1)} - \frac{6}{(x+5)(x-1)} = \frac{2(x-1)}{(x+5)(x+1)(x-1)} - \frac{6(x+1)}{(x+5)(x+1)(x-1)} = \frac{-4x+4}{(x+5)(x+1)(x-1)}$$

$$= \frac{-4(x-1)}{(x+5)(x+1)(x-1)} = \frac{-4}{(x+5)(x+1)}$$

Ex: $\frac{2x}{x+1} \cdot \frac{2x+2}{6x^2}$

$$= \frac{(2x)(2x+2)}{(x+1)(6x^2)}$$

$$= \frac{(2x)\,2(x+1)}{(x+1)\,6x^2}$$

$$= \frac{4x}{6x^2}$$

$$= \boxed{\frac{2}{3x}}$$

Ex $\frac{4x}{8x+2} \div \frac{6x+3}{8}$

$$= \frac{4x}{8x+2} \cdot \frac{8}{6x+3} = \frac{32x}{(8x+2)(6x+3)}$$

$$= \boxed{\frac{32x}{48x^2+36x+6}}$$

Lecture 12/1/23: LRB of rational functions

Recall a rational function is ~~the~~ a fraction of poly's.

Ex: $\dfrac{x^2+1}{x-1}$.

We can talk about their ~~\$~~ LRB and SRB's, like what we did for plain old polynomials.

<u>LRB</u>: ~~In the long~~ Let $f(x) = \dfrac{p(x)}{q(x)}$ be a ~~p~~ rational function w/ leading term $p(x) = ax^n$ and leading term of $q(x)$ bx^m. Then l.r.b of $\dfrac{p(x)}{q(x)}$ is the long run behavior of the power function

$$\frac{ax^n}{bx^m} = \frac{a}{b}x^{n-m}.$$

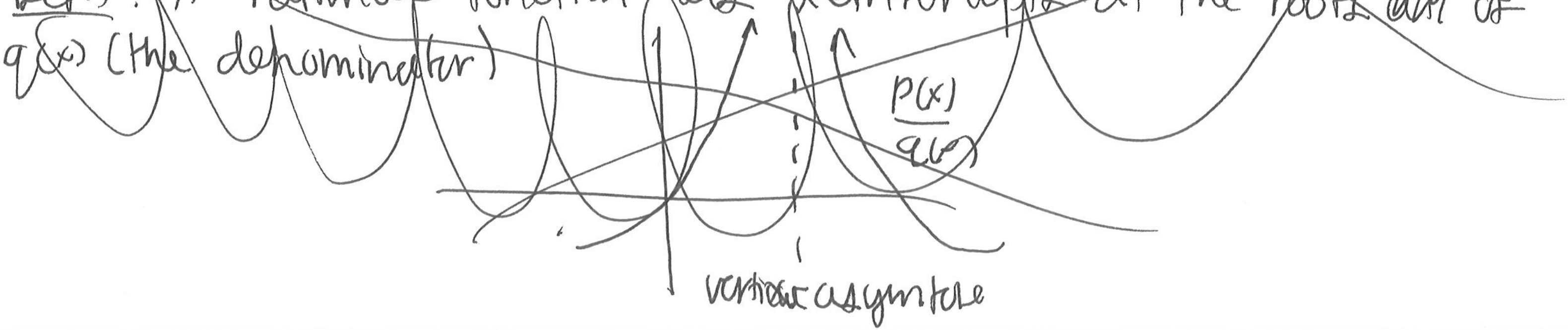
<u>Ex</u>: $f(x) = \dfrac{2x}{x^4-2x^3-1}$

① LRB $f(x)$ = LRB of $\dfrac{2x}{x^4} = 2x^{-3}$. Hence

$$\lim_{x\to\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 0.$$

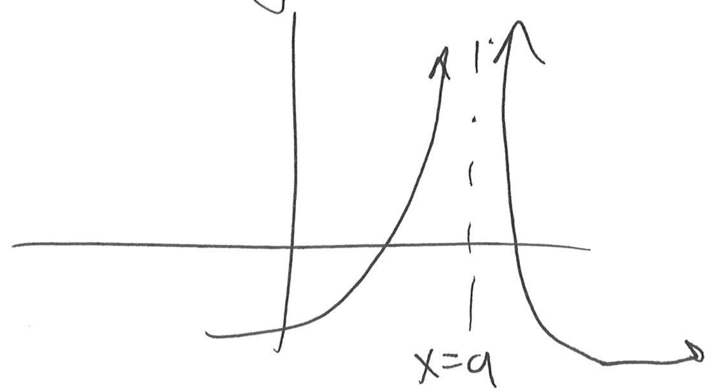
$f(x) = \dfrac{p(x)}{q(x)}$

<u>Defn</u>: A rational function has vertical asymptotes ~~x-intercepts~~ at the roots ~~of~~ of $q(x)$ (the denominator)

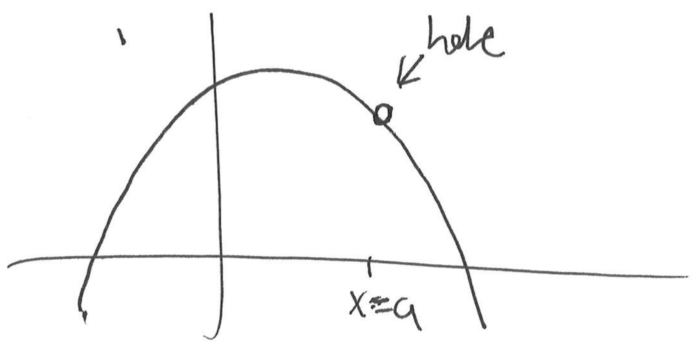


Lecture 12/2/23 : Short run behavior of ~~poly~~ rational functions.

Defn: $f(x) = \dfrac{p(x)}{q(x)}$. If a is a root of q (i.e. $q(a)=0$) and $p(a) \neq 0$ we say $f(x)$ has a vertical asymptote at $x = a$



Defn: $f(x) = \dfrac{p(x)}{q(x)}$. If a is a root of q (i.e. $q(a)=0$) and $p(a) = 0$ we say $f(x)$ has a hole at $x=a$.

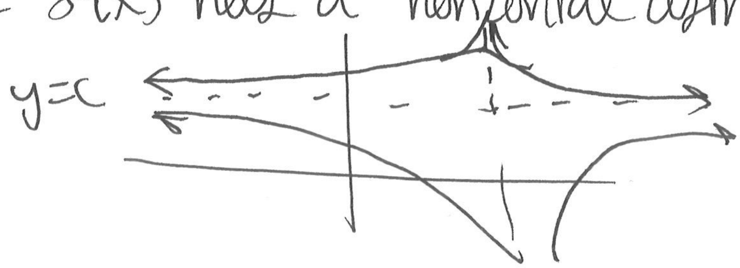


Defn: Let $f(x) = \dfrac{p(x)}{q(x)}$. If

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \underset{\neq \infty}{C} = \lim_{x\to -\infty} \frac{p(x)}{q(x)}$$

finite!

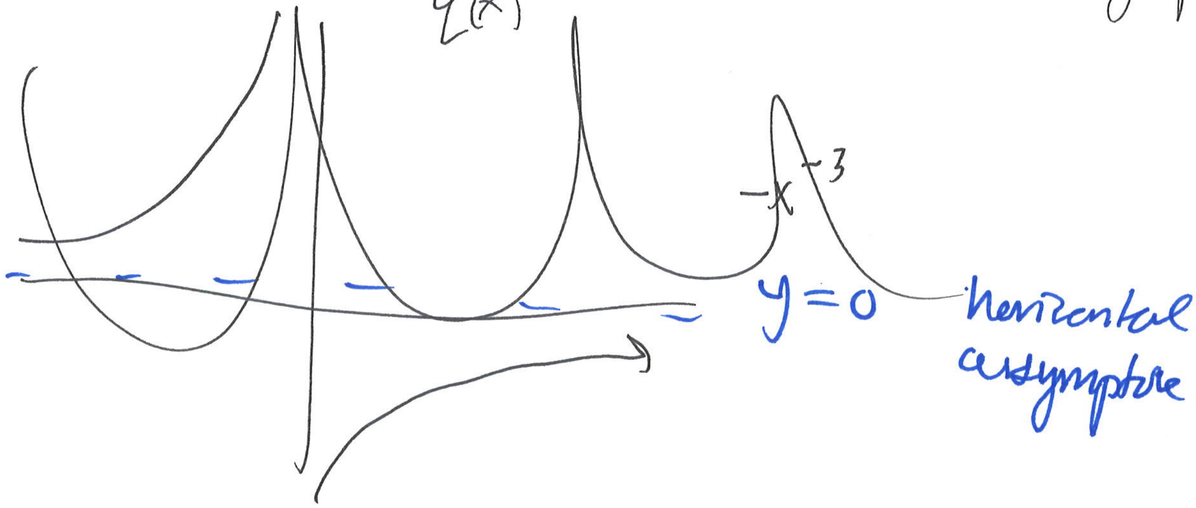
we say $f(x)$ has a horizontal asymptote at $y = C$



Defn: ~~Ashon~~ let $\frac{P(x)}{q(x)}$ be a rational function. If

$x \to \infty$

Both long run behaviors of $\frac{P(x)}{q(x)}$ are the same and ~~finite~~ equal to $c < \infty$. Then $\frac{P(x)}{q(x)}$ has a horizontal asymptote $y = c$



We will use all the ideas we have talked about to graph rational functions!

Graph

$$f(x) = \frac{(x+3)(x-2)}{x^2-1}$$

① Find zeros of $p(x)$ (numerator

$p(x) = (x+3)(x-2)$

zeros: $x=-3 \quad x=2$

② Find zeros of $q(x)$ and determine if they signify a hole or vertical int.

$q(x) = x^2-1 = (x+1)(x-1)$

roots: $x=1$, $x=-1$

- No holes since $p(x)$ and $q(x)$ don't share roots
- 2 vertical asymptotes at $x=1$ and $x=-1$.

③ LRB Divide leading terms of numerator and denominator

$$\lim_{x\to\infty} \frac{x^2}{x^2} = 1 = \lim_{x\to-\infty} \frac{x^2}{x^2}$$